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Analysis of Axisymmetric Functionally Graded Forced Vibrations due to Heat Sources in Viscothermoelastic Hollow Sphere using Series Solution

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Abstract. The aim of this article is to focus on forced vibrations due to thermal loading of functionally graded viscothermoelastic hollow sphere. The material has been chosen functionally graded due to easy exponent law. The power series solution of matrix Fröbenious method has been applied to solve mathematical model so developed to find temperature, displacement and stresses. The numerical solutions are obtained using MATLAB software for polymethyl methecrylate material. The convergence analysis have been applied to field functions by applying series solution, so that these field functions are uniformly convergent and the derived series have term by term differentiation. The computer simulated results for displacement, temperature change, radial stress and de–hoop stress have also been presented graphically. By using grading index the stress development, the stresses can be made compressive as well as tensile by taking different values of α positive or negative.

INTRODUCTION

Hunter et al. [1] and Flugge [2] used a lot of mathematical models, to provide somewhere to stay the energy dissipation in vibrating viscoelastic solids due to internal friction. Sharma [3] had investigated a model, based on infinite Kelvin–Voigt wave type in coupled viscothermoelastic plate. Othman et al. [4] have expanded the analysis in two–dimensional thermoviscoelastic medium in which he introduced two relaxation time parameters. Kanoria and Ghosh [5] studied the thermoelastic functionally graded relations in spherically isotropic hollow sphere in the context of generalized thermoelasticity. Bargi and Eslami [6] investigated the Green–Lindsay (GL) theory based on functionally graded thermoelastic hollow spheres. Alavi et al. [7] studied the functionally graded thermoelastic sphere subjected to mechanical and thermal loadings. Akimoto et al. [8] developed a technique by using Voigt model to compute the shear and bulk wave velocities of a viscoelastic sphere. Sharma et al. [9] investigated the stress free homogenous isotropic viscothermoelastic vibrations of hollow spheres using series solution. Akulenko and Nesterov [10] investigated the oscillations of in–homogenous rod based on elastic medium with inconsistent stiffness boundary conditions. Keles and Tutuncu [11] investigated the elastic cylinders (disks) and spheres in the context of free and forced elastic vibrations based on functionally graded materials. Sharma et al. [12] studied the free vibrations of axisymmetric functionally graded isotropic viscothermoelastic hollow sphere. Sharma et al. [13, 14] also studied the forced and free vibrations of functionally graded cylinders in the context of Lord and Shulman (LS) model of thermoelasticity. Sharma and Mishra [15] developed Lord and Shulman (LS) model of thermoelasticity in which the free vibrations of thermoelastic responses have been investigated due to functionally graded material of hollow sphere. In this paper the problem has been developed with non–classical theories of

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thermoelasticity developed by Lord and Shulman (LS) [16] and Green and Lindsay (GL) [17]. Neuringer [18] solved the indicial equation in which the complex roots have been applied and obtained by using series solution of Fröbenius method.

The purpose of this paper is to study analytically and numerically the exact forced vibrations due to heat sources of functionally graded generalized viscothermoelastic hollow sphere due to temperature input and temperature gradient. The analytically modeled equations have been investigated by using series solution of matrix Fröbenius method to obtain solutions for displacement, stresses and temperature. The computer analyzed results for polymethyl methecrylate material has been shown for graphical presentation.

FORMULATION OF THE PROBLEM

We consider a conducting viscothermoelastic hollow sphere of outer radius la and inner radius a at uniform temperature T_0 initially. The displacement in spherical coordinates (r, θ, φ) are stated as $u_\theta = u_\varphi = 0$ and $u_r = u(r, t)$ respectively. The governing equations of motion and heat conduction for homogenous isotropic hollow sphere of outer radius la and inner radius a in the absence of body forces and heat sources in the context of generalized viscothermoelasticity are taken by (Love [20] and Dhaliwal and Singh [21])

Stress–Displacement–Temperature Relation

$$\begin{pmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{pmatrix} = \begin{pmatrix} (\lambda^* + 2\mu^*) & 2\lambda^* \\ \lambda^* & 2(\lambda^* + \mu^*) \end{pmatrix} \begin{pmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{u}{r} \end{pmatrix} - \begin{pmatrix} \beta^* \left(1 + t_1 \delta_{2k} \dfrac{\partial}{\partial t}\right) T \\ \beta^* \left(1 + t_1 \delta_{2k} \dfrac{\partial}{\partial t}\right) T \end{pmatrix} \tag{1}$$

Equation of Motion

$$\frac{\partial \sigma_{rr}}{\partial r} = \rho \frac{\partial^2 u}{\partial t^2} - \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) \tag{2}$$

Equation of Heat Conduction

$$T_0 \beta^* \left(\frac{\partial}{\partial t} + t_0 \delta_{1k} \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K \frac{\partial T}{\partial r}\right) - \rho C_e \left(\frac{\partial}{\partial t} + t_0 \frac{\partial^2}{\partial t^2}\right) T \tag{3}$$

where $\lambda^* = \lambda_0 \left(1 + \alpha_0 \dfrac{\partial}{\partial t}\right)$, $\mu^* = \mu_0 \left(1 + \alpha_1 \dfrac{\partial}{\partial t}\right)$, $\beta^* = \beta_0^* \left(1 + \beta_0 \dfrac{\partial}{\partial t}\right)$,

Here σ_{ij}, $(i, j = r, \theta)$ is stress component; $u(r, t)$ and $T(\rho, \tau)$ are displacement and temperature respectively; K, ρ and C_e are thermal conductivity, mass density and specific heat at constant strain respectively; β^* and t_0, t_1 are viscothermoelastic coupling constant and thermal relaxation time parameters. Here the quantity δ_{ik}, $(i = 1, 2)$, is the Kronecker's delta in which $k = 1$ is taken for Lord-Shulman (LS) theory and $k = 2$ for Green-Lindsay (GL) theory. We assume the material to be functionally graded with power law in the sense that the density, thermal conductivity and modulus of elasticity differ with the radial coordinates accordingly

$$\lambda_0 = \lambda_e r^\alpha, \ \mu_0 = \mu_e r^\alpha, \quad \beta_0^* = \beta_e r^\alpha, \ \rho = \rho_e r^\alpha, \quad K = K_0 r^\alpha \tag{4}$$

where $\beta_e = (3\lambda_e + 2\mu_e)\alpha_T$, $\beta_0 = (3\lambda_e \alpha_0 + 2\mu_e \alpha_1)\alpha_T / \beta_e$ and α is the degree of in-homogeneity. The quantities α_0, α_1 are the mechanical relaxation times; λ_e, μ_e are Lame's parameters and α_T is the coefficient of linear thermal expansion of the material.

Initial and Regular Boundary Conditions

We consider in–homogenous functionally graded hollow sphere, initially at rest both mechanically and thermally, which is subjected to temperature input and temperature gradient at outer radius $r = la$ and inner radius $r = a$. The initial conditions can be written as

$$u(r\ ,\ t) = T(r\ ,\ t) = \frac{\partial T(r\ ,\ t)}{\partial t} = 0 \quad \text{at} \quad t = 0$$

The regular boundary conditions are stress free with temperature input and temperature gradient at the surfaces of the hollow sphere, can be written mathematically

$$\begin{aligned} &\sigma_{rr} = 0\ , && \text{at} \quad r = a\ , \quad la \quad \text{for} \quad t \ge 0 \\ &T = h_1 \exp(\text{-}i\Omega t)\ , && \text{at} \quad r = a\ , \quad la \quad \text{for} \quad t > 0 \end{aligned} \tag{5}$$

$$T_{,r} = h_2 \exp(\text{-}i\Omega t)\ , \quad \text{at} \quad r = a\ , \quad la \quad \text{for} \quad t > 0 \tag{6}$$

Here h_1 and h_2 are dimensionless constants and $\Omega = \frac{\omega a}{c_1}$, is the non–dimensional circular frequency of vibrations.

Solution of the Problem

To find the solution and remove the complexity of the equations, we introduce the following non–dimensional quantities:

$$\begin{aligned} &x = \frac{r}{a}\ , \ \tau = \frac{c_1 t}{a}\ , \ U = \frac{u}{a}\ , \ \theta = \frac{T}{T_0}\ , \ \varepsilon_T = \frac{T_0 \beta_e^2}{\rho_e C_e(\lambda_e + 2\mu_e)}\ , \ \bar{\varepsilon} = \frac{T_0 \beta_e}{(\lambda_e + 2\mu_e)}, \ \hat{\beta}_0 = \frac{c_1}{a}\beta_0\ ; \\ &\delta_0 = \hat{\alpha}_0 + 2\delta^2(\hat{\alpha}_1 - \hat{\alpha}_0)\ , \ \tau_0 = \frac{c_1}{a}t_0\ , \ \tau'_0 = \frac{c_1}{a}t_0\ , \ \tau_1 = \frac{c_1}{a}t_1\ , \ \hat{\alpha}_0 = \frac{c_1}{a}\alpha_0\ , \ \hat{\alpha}_1 = \frac{c_1}{a}\alpha\ ; \\ &\Omega^* = \frac{a\,\omega^*}{c_1}, \tau_{xx} = \frac{\sigma_{rr}}{\rho_e c_1^2}, \tau_{\theta\theta} = \frac{\sigma_{\theta\theta}}{\rho_e c_1^2}\ , \ \omega^* = \frac{C_e(\lambda_e + 2\mu_e)}{K_0}, \ c_1^2 = \frac{(\lambda_e + 2\mu_e)}{\rho_e}\ , \ c_2^2 = \frac{\mu_e}{\rho_e}, \ \delta^2 = \frac{c_2^2}{c_1^2} \end{aligned} \tag{7}$$

Using quantities (7) in equations (1) to (3) via equation (4) we get

$$\bar{\varepsilon}\left(1 + \hat{\beta}_0 \frac{\partial}{\partial \tau}\right)\left(1 + \tau_1 \delta_{2k} \frac{\partial}{\partial \tau}\right)\left(\frac{\partial \theta}{\partial x} + \frac{\alpha}{x}\theta\right) + \frac{\partial^2 U}{\partial \tau^2} = \left(1 + \delta_0 \frac{\partial}{\partial \tau}\right)\left(\frac{\partial^2 U}{\partial x^2} + \frac{m_1}{x}\frac{\partial U}{\partial x}\right) + \frac{\hat{m}_2}{x^2} U \tag{8}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{m_1}{x}\frac{\partial \theta}{\partial x} - \Omega^*\left(\frac{\partial}{\partial \tau} + \tau_0 \frac{\partial^2}{\partial \tau^2}\right)\theta = \varepsilon_T \frac{\Omega^*}{\bar{\varepsilon}}\left(1 + \hat{\beta}_0 \frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial \tau} + \tau'_0 \delta_{1k} \frac{\partial^2}{\partial \tau^2}\right)\left(\frac{\partial U}{\partial x} + \frac{2U}{x}\right) \tag{9}$$

$$\tau_{xx} = x^{\alpha}\left[2(1 - 2\delta^2)\left(1 + \hat{\alpha}_0 \frac{\partial}{\partial \tau}\right)\frac{U}{x} + \left(1 + \delta_0 \frac{\partial}{\partial \tau}\right)\frac{\partial U}{\partial x} - \bar{\varepsilon}\left(1 + \hat{\beta}_0 \frac{\partial}{\partial \tau}\right)\left(1 + \tau_1 \frac{\partial}{\partial \tau}\right)\theta\right] \tag{10}$$

$$\tau_{\theta\theta} = x^{\alpha}\left[\left(1 + \delta_0 \frac{\partial}{\partial \tau}\right)\frac{U}{x} + 2(1 - 2\delta^2)\left(1 + \hat{\alpha}_0 \frac{\partial}{\partial \tau}\right)\left(\frac{\partial}{\partial x} + \frac{1}{x}\right)U - \bar{\varepsilon}\left(1 + \hat{\beta}_0 \frac{\partial}{\partial \tau}\right)\left(1 + \tau_1 \frac{\partial}{\partial \tau}\right)\theta\right] \tag{11}$$

where

$$m_1 = \alpha + 2\ , \ \hat{m}_2 = 2\left[-\left(1 + \hat{\delta}_0 \frac{\partial}{\partial \tau}\right) + \left(1 + \hat{\alpha}_0 \frac{\partial}{\partial \tau}\right)(1 - 2\delta^2)\alpha\right] \tag{12}$$

We consider time harmonic vibrations [19] and define non–homogenous transformations by

$$\big(U(x)\ ,\ \theta(x)\big) = (w\ ,\ \Theta)\ x^{-\frac{1+\alpha}{2}} \exp(-i\Omega\tau) \tag{13}$$

where $\Omega = \frac{\omega\, a}{c_1}$,is the non dimensional circular frequency. Using these solutions (13) in equations (8) to (9) we obtain

$$\begin{bmatrix} \nabla^2 + \left(\frac{i\Omega}{\tilde{\delta}_0} - \frac{n^2}{x^2}\right) & \frac{\bar{\varepsilon}\, i\,\Omega\tilde{\beta}_0\,\tilde{\tau}_1}{\tilde{\delta}_0}\left(\frac{d}{d\,x} + \frac{(\alpha-1)}{2x}\right) \\ -i\frac{\varepsilon_T\Omega^*}{\bar{\varepsilon}}\Omega^3\,\tilde{\beta}_0\tilde{\tau}_0'\left(\frac{d}{d\,x} + \frac{3-\alpha}{2x}\right) & \nabla^2 + \left(\Omega^*\,\Omega^2\tilde{\tau}_0 - \left(\frac{1+\alpha}{2}\right)^2\frac{1}{x^2}\right) \end{bmatrix}\begin{bmatrix} w \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{14}$$

where

$$m_2 = 2\left(\frac{\alpha\tilde{\alpha}_0(1-2\delta^2)}{\tilde{\delta}_0} - 1\right),\; n^2 = \left(\frac{1+\alpha}{2}\right)^2 - m_2\;,\; \tilde{N}^2 = \frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right)\;,\; \tilde{\alpha}_0 = i\Omega^{-1} + \hat{\alpha}_0\;,$$

$$\tilde{\delta}_0 = i\Omega^{-1} + \delta_0\,,\; \tilde{\tau}_1 = i\Omega^{-1} + \tau_1\,\delta_{2k}\,,\tilde{\tau}_0 = i\Omega^{-1} + \tau_0\,,\; \tilde{\tau}_0' = i\Omega^{-1} + \tau_0'\delta_{1k}\; \tilde{\alpha}_1 = i\Omega^{-1} + \hat{\alpha}_1\;,\; \tilde{\beta}_0 = i\Omega^{-1} + \hat{\beta}_0\,,$$

Series Solution

Here in the matrix differential equation (14), $x = 0$ is regular–singular point, therefore there must be a solution of the form

$$Y = \sum_{k=0}^{\infty} Y_k\, x^{s+k} \tag{15}$$

where $Y = \begin{bmatrix} w & \Theta \end{bmatrix}'$, $Y_k = \begin{bmatrix} A_k & B_k \end{bmatrix}'$. Here s and A_k , B_k are the eigen value and unknown coefficients to be evaluated. The problem has been solved for $a \le r \le la$, $a > 0$ and the solution (15) is applicable in some interval $1 \le x \le R$, $R > 1$.

Substituting the assumed solution (15) in equations (14) and after simplification, we get

$$\sum_{k=0}^{\infty}\left(G_1(s+k)x^{-2} + G_2(s+k)x^{-1} + G\right)x^{s+k}Y_k = 0 \tag{16}$$

where $G_1(s+k) = \begin{bmatrix} (s+k)^2 - n^2) & 0 \\ 0 & \left((s+k)^2 - \left(\frac{1+\alpha}{2}\right)^2\right) \end{bmatrix}$;

$$G_2(s+k) = \begin{bmatrix} 0 & \frac{i\,\Omega\,\bar{\varepsilon}\,\tilde{\beta}_0\,\tilde{\tau}_1}{\tilde{\delta}_0}\left(s+k-\frac{\alpha-1}{2}\right) \\ -i\frac{\varepsilon_T\Omega^*}{\bar{\varepsilon}}\Omega^3 m_4\tilde{\beta}_0\tilde{\tau}_0'\left(s+k+\frac{3-\alpha}{2}\right) & 0 \end{bmatrix};\quad G = \begin{bmatrix} \frac{i\Omega}{\delta_0} & 0 \\ 0 & \Omega^*\Omega^2\tilde{\tau}_0 \end{bmatrix};$$

By equating to zero the coefficients of lowest power of x (i.e. x^{s-2}) in equation (16), we obtain the matrix equation:

$$\begin{bmatrix} (s^2 - n^2) & 0 \\ 0 & \left(s^2 - \left(\frac{1+\alpha}{2}\right)^2\right) \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = 0 \tag{17}$$

This is a non-trivial solution in system of matrix equation (17) will lead to the following indicial equation

$$s^4 - (n^2 + (\alpha^2 + 2\alpha + 1)/4)s^2 + n^2((\alpha+1)^2/4) = 0$$

The roots of indicial equation are given by

$$s_1 = n \;,\; s_2 = -n \quad s_3 = \left(\frac{1+\alpha}{2}\right), \quad s_4 = -\left(\frac{1+\alpha}{2}\right) \tag{18}$$

Here the roots s_1 and s_2 are complex and the roots s_3 and s_4 being real. The leading terms in the former case in the series solution (16) are of the type

$$(A_0 \quad B_0)' x^{s} = (A_0 \quad B_0)' x^{s_R} \left\{ \cos(s_I \log x) + i \sin(s_I \log x) \right\}$$

To obtain real solutions of the system of matrix differential equation (16), the sufficient use of any one of the complex root of the indicial equation is to find two independent real solutions of the system of matrix differential equations, see ref. Neuringer [18].

For this we have the choice of indicial roots, the system of equations (17) leads to

$$A_0(s_j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad B_0(s_j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{19}$$

Again equating to zero the next lowest coefficients degree term of x (i.e. x^{s-1}) in equation (16), we obtain

$$G_1(s_j + 1) Y_1 + G_2(s_j) Y_0 = 0 \tag{20}$$

On simplification the equation (20) provides us

$$Y_1 = -C_1 Y_0 \tag{21}$$

where $C_1 = \begin{bmatrix} 0 & c_{12}^{1}(s_j) \\ c_{21}^{1}(s_j) & 0 \end{bmatrix}$

Similarly the matrices of C_2 , C_3 , C_4 ....C_K can be calculated from the recurrence relation in next equation. The constants $c_{12}^{1}(s_j)$ and $c_{21}^{1}(s_j)$ are shown in appendix (A 1.1)

Again equating to zero the coefficients of like powers of x^{s+k}, we obtain

$$G_1(s_j + k + 2) Y_{k+2} + G_2(s_j + k + 1) Y_{k+1} + G\, Y_k = 0 \ , \quad k = 0, 1, 2, 3, ... \tag{22}$$

On simplification the equation (22), we get

$$Y_{k+2} = -\begin{bmatrix} 0 & G_{12}^{k}(s_j) \\ G_{21}^{k}(s_j) & 0 \end{bmatrix} Y_{k+1} - \begin{bmatrix} G_{11}^{k}(s_j) & 0 \\ 0 & G_{22}^{k}(s_j) \end{bmatrix} Y_k \quad , \quad k = 0, 1, 2, 3, ... \tag{23}$$

The constants $G_{11}^{k}(s_j)$, $G_{12}^{k}(s_j)$, $G_{21}^{k}(s_j)$ and $G_{22}^{k}(s_j)$ are shown in appendix (A 1.2) to (A 1.3). Putting $k = 0, 1 \ , 2 \ , 3 \ ...$ we get required reoccurrence relation in equation (23). It can be easily shown by continuing in like manner that the matrices $C_{2k}(s_j)$ have same form as that of $G_1(s+k)$ and the matrices $C_{2k+1}(s_j)$ have alike $G_2(s+k)$ form. Thus, in general, we have

$$\left(Y_{2k}(s_j) \ , \ Y_{2k+1}(s_j) \right) = \left(C_{2k}(s_j) \ , \ -C_{2k+1}(s_j) \right) Y_0 \ , \ k = 1 \ \ 2 \ , 3 \ ... \tag{24}$$

$$\text{where } C_{2k}(s_j) = \begin{bmatrix} c_{11}^{2k}(s_j) & 0 \\ 0 & c_{22}^{2k}(s_j) \end{bmatrix} \quad ; \quad C_{2k+1}(s_j) = \begin{bmatrix} 0 & c_{12}^{2k+1}(s_j) \\ c_{21}^{2k+1}(s_j) & 0 \end{bmatrix} \tag{25}$$

The values of $c_{11}^{2k}(s_j)$, $c_{22}^{2k}(s_j)$, $c_{12}^{2k+1}(s_j)$ and $c_{21}^{2k+1}(s_j)$ are shown in Appendix (A 1.4) to (A 1.8). From equations (25), it can be seen that

$$C_{2k}(s_j) \approx O\left(k^{-1}\right) C^{*} \qquad , \qquad C_{2k+1}(s_j) \approx O\left(k^{-1}\right) C^{**} \tag{26}$$

where C^{*} and C^{**} are defined in Appendix (A 1.9). Due to Cullen [22], in complex field a sequence of matrix $\{B_k\}$ converges, as $(\lim_{k \to \infty} B_k = B)$, if every value of the k^2 component the sequence converges. Using above facts, we easily conclude that both the matrices $C_{2k}(s_j) \to 0$ and $C_{2k+1}(s_j) \to 0$, as $k \to \infty$. This means that the series in equation (15) are uniformly and absolutely convergent. Thus the derived series might be term by term differentiation and are analytic functions.

Formal Solution for Displacement, Temperature and Stress

With the help of equations (24) and (25), the general solution (13) becomes

$$U(x,\tau)=\sum_{k=0}^{\infty}\left[\sum_{j=1}^{2}E_{jk}c_{11}^{2k}(s_j)x^{s_j}-\sum_{j=3}^{4}E_{jk}c_{12}^{2k+1}(s_j)x^{1+s_j}\right]x^{2k-\frac{1+\alpha}{2}}\exp(-i\Omega\tau)$$

$$\theta(x,\tau)=\sum_{k=0}^{\infty}\left[-\sum_{j=1}^{2}E_{jk}c_{21}^{2k+1}(s_j)x^{1+s_j}+\sum_{j=3}^{4}E_{jk}c_{22}^{2k}(s_j)x^{s_j}\right]x^{2k-\frac{1+\alpha}{2}}\exp(-i\Omega\tau) \tag{27}$$

where E_{1k} , E_{2k} , E_{3k} , E_{4k} are arbitrary constants to be evaluated. With the help of equations (27), the stress and temperature gradient are achieved as

$$\tau_{xx}=-i\Omega\tilde{\delta}_0\sum_{k=0}^{\infty}\left[\begin{array}{l}\sum_{j=1}^{2}\left\{\left(\left(R_{11}^{1}+s_j+d^*\right)c_{11}^{2k}(s_j)-g^*x^2c_{21}^{2k+1}(s_j)\right)E_{jk}\right\}\\-\sum_{j=3}^{4}\left\{\left(\left(R_{12}^{2}+s_j+d^*\right)c_{12}^{2k+1}(s_j)-g^*c_{22}^{2k}(s_j)\right)E_{jk}x\right\}\end{array}\right]x^{2k+s_j-\frac{1+\alpha}{2}}\exp(-i\Omega\tau)$$

$$\frac{d\theta}{dx}=\sum_{k=0}^{\infty}\left[-\sum_{j=1}^{2}\left(R_{12}^{2}+s_j\right)E_{jk}c_{21}^{2k+1}(s_j)+\sum_{j=3}^{4}\left(R_{11}^{1}+s_j\right)\frac{1}{x}E_{jk}c_{22}^{2k}(s_j)\right]x^{2k+s_j-\frac{1+\alpha}{2}}\exp(-i\Omega\tau) \tag{28}$$

where $R_{11}^{1}=\left(2k-\dfrac{1+\alpha}{2}\right)$, $R_{12}^{2}=\left(2k+\dfrac{1-\alpha}{2}\right)$, $d^*=\dfrac{2(1-2\delta^2)\tilde{\alpha}_0}{\tilde{\delta}_0}$, $g^*=\dfrac{i\Omega\bar{\varepsilon}\tilde{\beta}_0\tilde{\tau}_1}{\tilde{\delta}_0}$;

The equations (27) and (28) represent the proper solution of the problem and field functions, which are obtained from the boundary conditions (5)–(6) for the purpose of the solution that concentrate the considered situation. Upon using the boundary conditions from equations (5)–(6) after non–dimensionalized form of equations for the domain $1\le x\le l$ we obtain

$$\left(\begin{array}{l}\sum_{j=1}^{2}\left\{\left(\left(R_{11}^{1}+s_j+d^*\right)c_{11}^{2k}(s_j)-g^*x^2c_{21}^{2k+1}(s_j)\right)E_{jk}\right\}-\\\qquad\sum_{j=3}^{4}\left\{\left(\left(R_{12}^{2}+s_j+d^*\right)c_{12}^{2k+1}(s_j)-g^*c_{22}^{2k}(s_j)\right)E_{jk}x\right\}\end{array}\right)x^{2k+s_j-\frac{1+\alpha}{2}}e^{-i\Omega\tau}=0 \tag{29}$$

$$\left(-\sum_{j=1}^{2}E_{jk}c_{21}^{2k+1}(s_j)x^{1+s_j}+\sum_{j=3}^{4}E_{jk}c_{22}^{2k}(s_j)x^{s_j}-h_1\right)x^{2k-\frac{1+\alpha}{2}}e^{-i\Omega\tau}=0 \tag{30}$$

$$\left(-\sum_{j=1}^{2}\left(R_{12}^{2}+s_j\right)E_{jk}c_{21}^{2k+1}(s_j)+\sum_{j=3}^{4}\left(R_{11}^{1}+s_j\right)\frac{1}{x}E_{jk}c_{22}^{2k}(s_j)-h_2\right)x^{2k+s_j-\frac{1+\alpha}{2}}e^{-i\Omega\tau}=0 \tag{31}$$

NUMERICAL RESULTS AND DISCUSSION

For numerical computations the MATLAB software has been carried out for temperature gradient of thermally loaded sphere. For numerical computations the polymethyl methacrylate material has taken for which the physical data is given below [4]:

$\omega^*=1.11\times10^{11}\,s^{-1}$, $T_0=773$ K , $\delta^2=0.333$, $\hat{\alpha}_0=\hat{\alpha}_1=0.05$, $\tau_0=0.02$, $\varepsilon_T=0.045$,

$\tau_1=0.03$, $\rho=1190\ \mathrm{kg\,m^{-3}}$, $C_e=1400\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $K=0.19\ \mathrm{W\,m^{-1}\,K^{-1}}$, $\alpha_T=77\times10^{-6}\ \mathrm{K^{-1}}$

The numerical results have been taken from equations (29) – (31) up to six decimal places by choosing appropriate value of $(k=20)$ in order to get the results in desired accuracy.

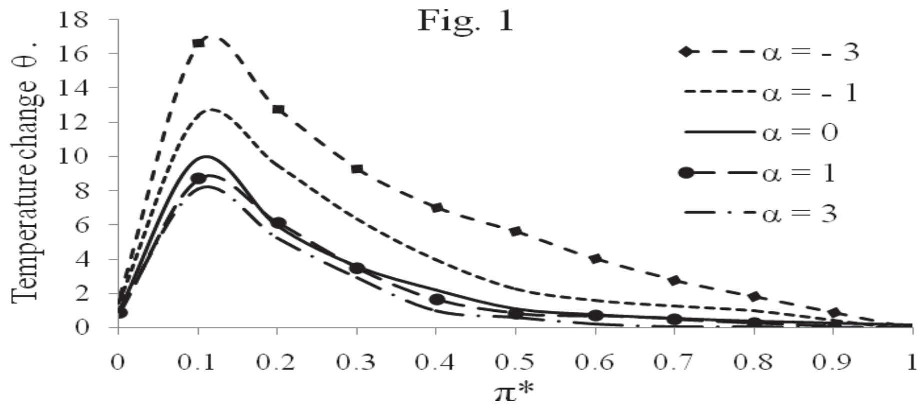

FIGURE 1. Temperature (θ) versus normalized thickness (π^*)

Figs. 1 to 4 show the change of temperature (Θ), displacement (U), radial stress (τ_{xx}) and de–hoop stress $(\tau_{\theta\theta})$ against normalized thickness (π^*) which is defined as $0 \le \pi^* \le 1$, where $\left(\pi^* = (x-1)/(l-1)\right)$ for $\alpha = $ -3.0, -1.0, 0.0, 1.0, 3.0 in case of viscothermoelastic hollow sphere. It is observed that the temperature change initially increases up to $\alpha = 0.1$ after that it decreases with increasing values of π^*.

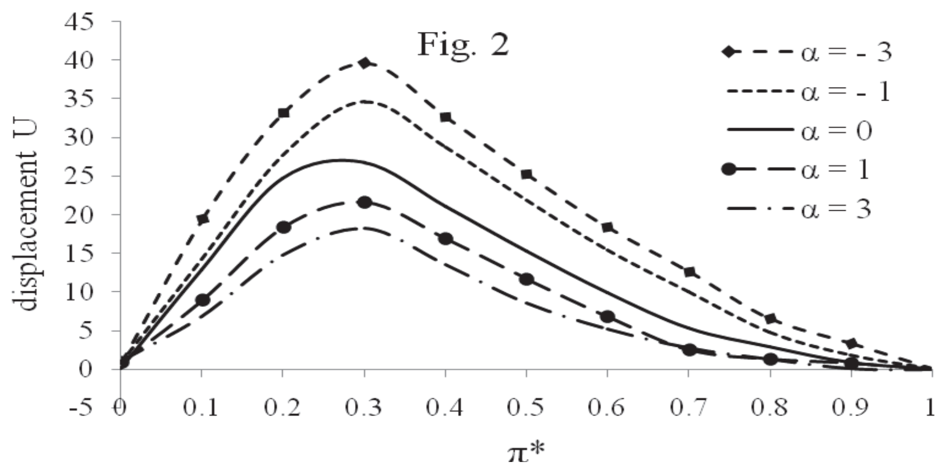

FIGURE 2. Displacement (U) versus normalized thickness (π^*)

This is noticed from Fig. 2 that the displacement (U) initially increases and become maximum at $0.2 \le \alpha \le 0.35$, after that it decreases with increasing values of π^*. Fig. 3 and Fig. 4 have plotted for radial stress (τ_{xx}) and de–hoop stress $(\tau_{\theta\theta})$ versus normalized thickness (π^*). It is revealed from Fig. 3 that the radial stress has been noticed high at $\alpha = 0.1$ which is positive for $\alpha = $ -3.0, -1.0, 0.0 and negative for $\alpha = 1.0$, 3.0. Fig. 4 depicts that the value of de–hoop stress $(\tau_{\theta\theta})$ is maximum at $\alpha = 3.0$, 1.0 i.e. positive and minimum i.e. negative for $\alpha = $ -3.0, -1.0, 0.0. Effect of variation of radial stress (τ_{xx}) is apposite to that de–hoop stress $(\tau_{\theta\theta})$. With increase in the value of normalized thickness (π^*) the variation of vibrations of radial stress (τ_{xx}) and de–hoop stress $(\tau_{\theta\theta})$ become asymptotic at $(\pi^*$ ³ $0.9)$.

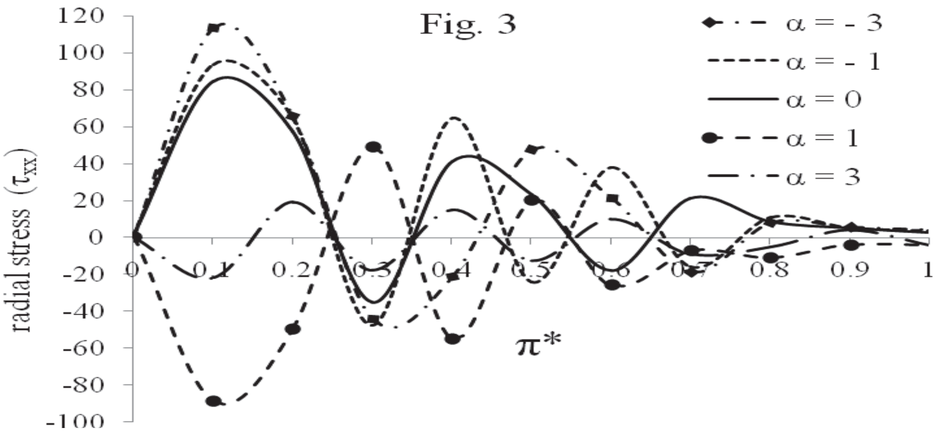

FIGURE 3. Radial stress (τ_{xx}) versus normalized thickness (π^{*})

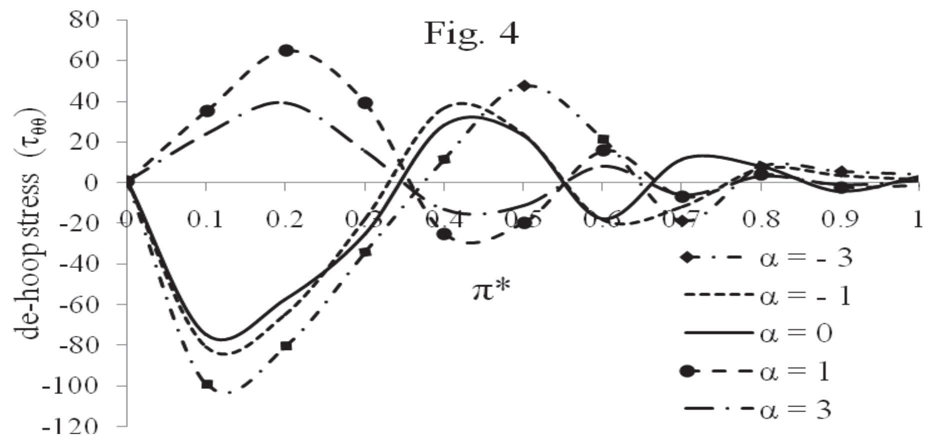

FIGURE 4. De–hoop stress $(\tau_{\theta\theta})$ versus normalized thickness (π^{*})

The magnitudes of temperature change, displacement and stresses are large in case of $\alpha = -3.0$ and has small at magnitude $\alpha = 3.0$ as compare to different values of (α) which clearly indicates the effect of in–homogeneity of the material. The numerical results are consistent with Sharma and Mishra [15] in the absence of viscous effects. The stress development in Fig. 3 is tensile in the case of $\alpha = -3.0$, -1.0 , 0.0 and compressive for $\alpha = 1.0$, 3.0 and reverse behaviour in stress development of Fig. 4 have been occurred which clearly show the positive and negative effect of in–homogeneity of the material.

CONCLUSION

The matrix Fröbenius method of power series solution has been applied successfully to investigate the forced vibrations due to heat sources of viscothermoelastic hollow sphere. The field functions i.e. temperature, displacement and stresses in the context of generalized theories assumed in this paper are continuous. The closed form solution for fields functions have been derived for axisymmetric viscothermoelastic spheres. Convergence of the series solution has clearly indicated that derived series are term by term differentiated and analytic. This is noticed that with the help of grading index, the behaviour of stress development can be made tensile as well as compressive or vice-versa and deformation can be checked (increased / decreased) as per requirement. The magnitude of all field functions are large at $\alpha = -\,0.3$ and small magnitude at $\alpha = 3.0$ as compare to different values of (α) which clearly indicates the effect of in–homogeneity index of the material.

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