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Quantifying matrix product state

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Abstract Motivated by the concept of quantum finite-state machines, we have investigated their relation with matrix product state of quantum spin systems. Matrix product states play a crucial role in the context of quantum information processing and are considered as a valuable asset for quantum information and communication purpose. It is an effective way to represent states of entangled systems. In this paper, we have designed quantum finite-state machines of one-dimensional matrix product state representations for quantum spin systems.

Keywords Quantum finite-state machines · Matrix product state · Quantum spin system · AKLT model · GHZ model · W state · Cluster state · Entanglement

Abbreviations

TN	Tensor network
TNS	Tensor network state
MPS	Matrix product state
PEPS	Projected entangled pair state
TTN	Tree tensor network
MERA	Multiscale entanglement renormalization ansatz
FSM	Finite-state machine
SFSM	Stochastic finite-state machine
QFSM	Quantum finite-state machine

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1 Introduction

The simulation of quantum many-body systems on a classical computer is a difficult task because the dimension of Hilbert space increases exponentially with the size of the system. But, the development of quantum simulators allows us to simulate the dynamics of interacting quantum many-body systems. In 1982, Feynman [1] initially proposed the idea of quantum computing after performing a quantum mechanics simulation on a classical computer and observed that n qubits can simulate n spin-1/2 particles on a quantum computer. The most significant property '*entanglement*' separates the classical world from the quantum world. It is one of the most central topics in quantum information theory. Quantum entanglement is purely quantum mechanical correlation between two parts of the quantum system. In quantum many-body systems, it provides a new way to describe the correlation between two particles.

In recent years, tensor network theory has become increasingly popular. Tensor network states are a new language, based on entanglement, for quantum many-body systems. It is widely used to simulate strongly entangled correlated systems [2]. The DNA is the fundamental building block of a person. Similarly, the tensor is the fundamental building block of a quantum state. Therefore, the tensor is the DNA of the wave function in which properties of the quantum many-body states can be read from the individual small tensors alone. Such a structure is called a tensor network (TN). Such a structure includes more degrees of freedom to attach the different tensors together. These extra degrees of freedom connecting indices between the tensors are called bond indices, and lines which do not connect one tensor to other are called open indices. Figure 1 shows the TN diagrams:

Tensor network states can be classified on the basis of dimensions along which the tensors are traversed. In 1993, White [3] introduced *density matrix renormalization group (DMRG)*, a most common TN method for simulation of one-dimensional strongly correlated quantum systems. It is based on most well-designed class of TN states called *matrix product state (MPS)*. In the last two decades, DMRG is considered as a method of reference to study the stationary properties of one-dimensional strongly correlated quantum systems. MPS provides an efficient approximation of realistic local Hamiltonians and can be generated by sequential generation of tensors.

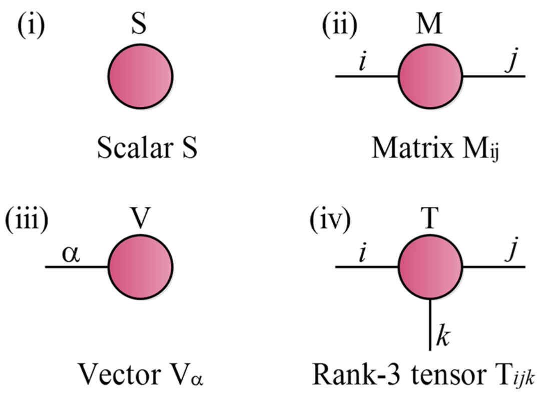

Fig. 1 Tensor networks: (**i**) scalar, (**ii**) matrix, (**iii**) vector, (**iv**) rank-3 tensor

Regardless of success, there are restrictions remaining in the dimensions, and classes of Hamiltonians that can be simulated with MPS-based methods. To overcome such restrictions, various new algorithms have been proposed based on different types of TN states. The first one was the *projected entangled pair states (PEPS)*, which is a generalization of MPS to two and higher dimensions, *tree tensor network (TTN)*, which has a real space renormalization group structure and has tree-like structures with no loops, and the *multiscale entanglement renormalization ansatz (MERA*), which removes a short range entanglement [4].

2 Preliminaries

In this section, we review some formal definitions and related properties that will be used in this paper.

Definition 1 [5] A finite-state machine (FSM) is defined as a quintuple $(Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

- Q is a finite set of states,
- Σ is finite set of alphabets,
- δ is a transition function: $Q \times \Sigma \rightarrow Q$,
- $q_0 \in Q$ is an initial state,
- $F \subseteq Q$ is a set of final states.

Definition 2 [6,7] A stochastic finite-state machine (SFSM) is defined as a triple $(S,\ X,\ Y, T^{(y)} : y \in Y)$, where

- S is a finite set of states,
- X and Y are input and input alphabets, respectively,
- $T^{(y)}$ are sub-stochastic matrices such that $T = \sum_{y \in Y} T^{(y)}$, where T is a stochastic matrix and $T^{(y)}$ is probability of transition from one state to other.

To process the languages with SFSM, we adopt Dirac's bra–ket notation. It is closed under addition and multiplication by scalars. Using Dirac's notation, the vectors are denoted by kets $|u\rangle$ (row vectors). We can associate with each ket a vector in the dual space called bra $\langle v|$ (column vectors). In SFSM, the probability associated with words $w_1 w_2 w_3 \ldots w_L \in Y^L$ of length L is computed as

$$\Pr\,(w_L) = \langle \pi | T^{(w_L)} | \eta \rangle, \tag{2.1}$$

where $\langle \pi |$ is stationary probability density such that $\langle \pi | = (\pi_1,\ \pi_2,\ \pi_3, \ldots, \pi_S)$ satisfy $\pi_1 \geq 0$; $\sum_i^S \pi_i = 1$; $T^{(w_L)} = T^{(w_1)} T^{(w_2)} T^{(w_3)} \ldots\ T^{(w_L)}$ are transition matrices; and $|\eta\rangle = (1,\ 1,\ 1, \ldots, 1)^T$ is a column vector with $|S|$ components [6,7]. We define a QFSM that relates to the standard quantum mechanical explanation of a physical experiment.

Definition 3 [6,7] A quantum finite-state machine (QFSM) is defined as a quintuple $\left(Q,\ \langle\Psi|, X, Y, P^{(y)} : y \in Y, U(y)\right)$,where

- Q is a finite set of states,
- $\langle\Psi|$ is a state vector which belongs to n-dimensional Hilbert space H,
- X and Y are input and output alphabets, respectively,
- $P^{(y)}$ is a mutually orthogonal projection operator such that $\sum_{y \in Y} P^{(y)} = 1$,
- $U(y)$ is a transition matrix such that $U(y) = U.P^{(y)}$, where U is an unitary matrix and $P^{(y)}$ is an orthogonal projection operator.

A quantum deterministic finite-state machine is a quantum finite-state machine in which each matrix $U(y)$ has at most one nonzero entry per row. A QFSM is said to be quantum transducer [7] with $|X| = 1$. In QFSM, there exists a basis vector v_i for each quantum state $q_i \in Q$, having single nonzero entry of a one at ith coordinate. The set of basis vectors span the Hilbert space H. If a state $q_i \in Q$ having incoming transition which outputs symbol y_1, then $P^{(y_1)} v_i = v_i$ [6]. Consider another incoming transition that outputs symbol y_2, and then $P^{(y_2)} v_i = v_i$. Subsequent to $y_1 \neq y_2$, by mutual orthogonal of projections, it satisfies that $P^{(y_1)} P^{(y_2)} = 0$.

To measure the probability of words for QFSMs, we define density operator ρ on a finite set of states that satisfy that $\{\Psi_1, \ldots, \Psi_k\}$ as

$$\rho = \sum_{i}^{k} \rho_i \, |\Psi_i\rangle \, \langle\Psi_i|, \tag{2.2}$$

where ρ_i is the probability for the system in the state of Ψ_i, and Ψ_i's are the diagonal basis for ρ. The density operator on Hilbert space must satisfy the trace condition, i.e., $Tr\,(\rho) = 1 \,|\rho \geq 0$, where $Tr(\ )$ refers to the sum of the diagonal elements of matrices. Similar to stationary density operator of SFSMs, the stationary density operator ρ for QFSMs is defined as

$$\rho = \sum_{y \in Y} P^{(y)} U^* \rho \, U P^{(y)}. \tag{2.3}$$

It is invariant under unitary evolution. The stationary density operator of QFSM is $\rho = |Q|^{-1} \cdot\ 1$ (proved in [6]). In QFSM, the probability of a single character $y \in Y$ depends on dimensions of projection operators and computed as

$$\Pr\,(y) = |Q|^{-1} \cdot\ \dim\left(P^{(y)}\right). \tag{2.4}$$

The probability associated with words $w_1 w_2 w_3 \ldots w_L\ \in Y^L$ of length L is computed as

$$\Pr\,(w_L) = Tr\left(U^*(w_L) \rho U(w_L)\right), \tag{2.5}$$

where $\rho = |Q|^{-1} \cdot\ 1$ is stationary density operator and $U\,(w_L) = U(w_1)\ U(w_2)\ U(w_3)\ \ldots\ U(w_L)$.

Definition 4 [6] A square matrix A of size n is doubly stochastic (or bistochastic) if all its entries are nonnegative real numbers and each of its rows and columns sums to 1. It is unistochastic if there exists a unitary matrix U such that $A_{ij} = |U_{ij}|^2$.

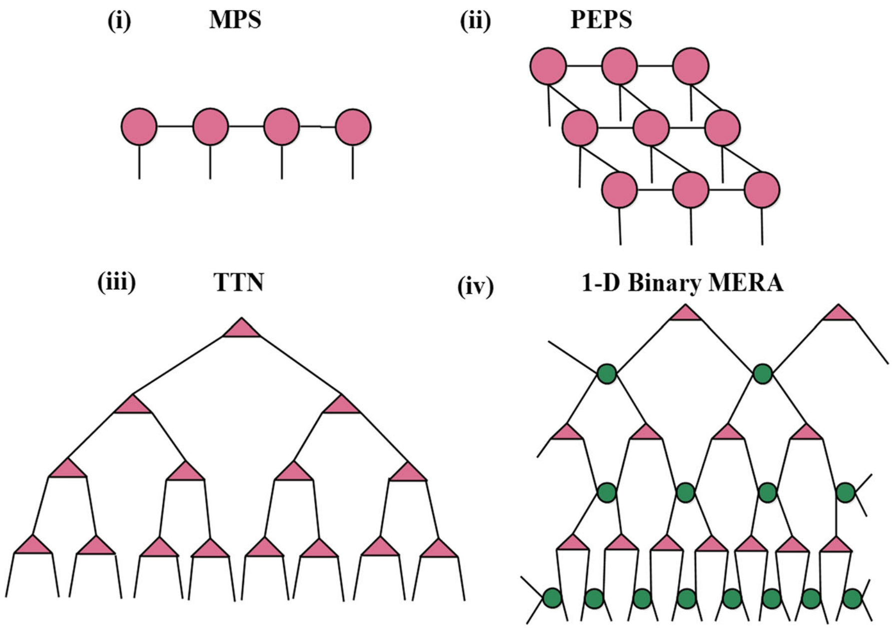

Fig. 2 Tensor network states: (**i**) MPS, (**ii**) PEPS, (**iii**) TTN, (**iv**) 1-D binary MERA

3 Matrix product state

The family of MPS is the most prominent example of TN states. There are various powerful methods such as density matrix renormalization group (DMRG) algorithm, power wave function renormalization group (PWFRG) and time-evolving block decimation (TEBD) based on MPS to simulate one-dimensional quantum many-body systems [8]. Figure 2(i) shows the MPS as one-dimensional array of tensors and an example of finite system of 4 sites. There exists one tensor for each site in quantum many-body systems. MPS shows a certain geometry that is relevant to experimental and quantum information theoretic applications. The family of MPS consists of following non-trivial states:

3.1 GHZ state

A Greenberger–Horne–Zeilinger (GHZ) state is an entangled quantum state [2,10] which has many non-classical properties. The GHZ state of N-spins 1/2 is defined as

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right). \tag{3.1}$$

It can be represented independently by the matrices:

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.2}$$

The GHZ state for 3-qubit is $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. It is highly entangled quantum state of N-spins, which has some non-trivial entanglement properties [11].

3.2 AKLT state

In 1987, Affleck–Kennedy–Lieb–Tasaki [12] state introduced an extension of quantum Heisenberg spin model. It is one of the most interesting quantum states in correlation physics, which is a ground state of quantum spin chain of spin-1. It is given by the Hamiltonian:

$$H = \sum_i \left(S_i \cdot S_{i+1} + \frac{1}{3} (S_i \cdot S_{i+1})^2 \right). \tag{3.3}$$

In MPS representation of AKLT state, each spin-1 is replaced by a pair of symmetrized spin-1/2. Consider triplet states represented as spin-1 states

$$\begin{aligned} |+\rangle &= |\uparrow\uparrow\rangle, \\ |0\rangle &= \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}, \\ |-\rangle &= |\downarrow\downarrow\rangle, \end{aligned}$$

whereas adjacent pairs of spin-1/2 are linked in a singlet state such as $\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$. The normalized MPS representation of N-chained AKLT state is:

$$|\psi\rangle = \sum_{\sigma}^{N} Tr\left[A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_N}\right] |\sigma_1 \sigma_2 \ldots \sigma_N\rangle. \tag{3.4}$$

Following are the three matrices categorized with Pauli matrices indices as $\sigma_i{}'$s [3].

$$A^+ = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix}, \quad A^0 = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad A^- = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}. \tag{3.5}$$

It can be computed as

$$\begin{aligned} \langle\psi|\psi\rangle &= \sum_{\sigma_i, \sigma_i{}'}^{N} \langle\sigma|\sigma'\rangle\, Tr\left[A^{\sigma_1{}'} A^{\sigma_2{}'} \cdots A^{\sigma_N{}'}\right] Tr\left[A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_N}\right] \\ &= Tr\left(\sum_{\sigma_1} A^{\sigma_1 *} \otimes A^{\sigma_1}\right) \ldots \left(\sum_{\sigma_1} A^{\sigma_N *} \otimes A^{\sigma_N}\right) \\ &= Tr\ E^N, \end{aligned} \tag{3.6}$$

where $E = \sum_{\sigma} A^{\sigma *} \otimes A^{\sigma} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}$.

3.3 Cluster state

In 2009, Raussendorf, Browne and Briegel [13] introduced the concept of cluster state. It is a type of highly entangled state multiple qubits [2]. Cluster states are the unique ground state of the 3-body interactions as $\sum_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z$. Consider a cluster state of 2-qubits:

$$\begin{aligned} |\phi_2\rangle &= |+\rangle_1 |+\rangle_2 = \frac{1}{2}\left(|0\rangle + |1\rangle\right)\left(|0\rangle + |1\rangle\right) \rightarrow \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle\right) \\ &= \frac{1}{2}\left(\left(|0\rangle + |1\rangle\right)|0\rangle\left(|0\rangle - |1\rangle\right)|1\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\left(|+\rangle_1|0\rangle_2\right) + \left(|-\rangle_1|1\rangle_2\right)\right). \end{aligned} \tag{3.7}$$

It forms Bell state. Its matrix product state is represented as

$$A^0 = |0\rangle|+\rangle = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^1 = |1\rangle|-\rangle = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}. \tag{3.8}$$

3.4 W state

$$|W\rangle = \frac{1}{\sqrt{n}}\left(|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle\right). \tag{3.9}$$

W state is an entangled quantum state represents and refers to the quantum superposition of pure states with equal coefficients [11]. It represents specific type of multipartite entanglement in which exactly one of the qubits is in excited state $|1\rangle$, while all others are in ground state $|0\rangle$. The W state for 3-qubit is $\frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$. It is represented by the matrices [9]:

$$\begin{aligned} A(1)^0 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(2)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A(3)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ A(1)^1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(2)^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A(3)^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.10}$$

Both $|GHZ\rangle$ and $|W\rangle$ represent two different kinds of tripartite entanglement. $|W\rangle$ is less entangled than the $|GHZ\rangle$, because it leaves bipartite entanglements on measure one of its sub-systems. But on measurement of $|GHZ\rangle$, it collapse into a mixture or a pure state.

4 Constructing QFSM

In this section, we have designed QFSM of various MPS models and proved that it is equivalent to MPS representation of ground state of quantum spin systems. It is true that every QFSM generates a SFSM with the same word probabilities. Let $M_Q = \big(Q,\ \langle\Psi|\ , X,\ \ Y,\ P^{(y)}:\ y \in Y,\ U(y)\big)$ be a QFSM and $M_S' = (S,\ \ X,\ \ Y,\ \ T^{(y)'}:\ y \in y)$ be a SFSM, where $|Q| = |S|$ and $T_{ij}^{(y)'} = |U(y)_{ij}|^2$ (proved in [6]) from the definition of equivalent SFSM. Therefore, M_S' produces same probabilities for every word as M_Q.

Consider a one-dimensional matrix product state; our process is to construct a quantum finite-state machine as follows: We have equivalent matrix product state models with unitary matrix or with unistochastic transition matrix T. Then, we construct a quantum finite-state machine with the same number of states. It is proved that every deterministic quantum generator has an equivalent deterministic classical generator that produces the same stochastic process if there is a unitary evolution and projective measurement for quantum process [14,15]. Amanda constructed quantized versions of several well-known stochastic finite-state machines [6]. There are two ways to construct quantum finite-state machine from a classical machine: We have unistochastic matrix for which there exists a unitary matrix such that $T_{ij} = |U_{ij}|^2$, and other using quantum analogy $U(y) = U \cdot P^{(y)}$ for the symbols transition matrix and projection operators. Following are the construction of QFSM of various MPS models:

4.1 GHZ state

In the construction of QFSM, we need to find a unitary matrix U for which $T_{ij} = |U_{ij}|^2$. From the definition of QFSM, recall the $U(y) = U \cdot P^{(y)}$, which is quantum analogy of the symbol-labelled transition matrices. We consider the following quantum finite-state machine for the maximum entangled GHZ state of 3-qubit:

$$Q = \{A,\ B\},\quad Y = \{0,\ 1\},\quad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},\quad P^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$P^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

The transition matrices are:

$$U(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},\quad U(1) = \tfrac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Correspondingly, the stochastic finite-state machine of GHZ state is defined as:

$$S = \{A, B\},\quad Y = \{0,\ 1\},\quad T^{(0)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\quad T^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

It can be easily checked that GHZ state has a unistochastic matrix $T = T^{(0)} + T^{(1)}$. The matrix T satisfies $T_{ij} = |U_{ij}|^2$, where U is unitary matrix (Fig. 3). The probability of words for the GHZ state with words up to length $L = 3$ is given in Table 1.

Thus, the quantized version of the GHZ state is shown in Fig. 4. It can easily be verified that the probabilities of words produced by the above QFSM and SFSM for GHZ state are same. Therefore, QFSM and SFSM for GHZ state are equivalent (Table 2).

4.2 AKLT state

The two-state stochastic finite-state machine of AKLT is given as

$$S = \{A, B\}, \quad Y = \{0, 1\}, \quad T^{(+)} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} \\ 0 & 0 \end{pmatrix},$$
$$T^{(0)} = \begin{pmatrix} -1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix}, \quad T^{(-)} = \begin{pmatrix} 0 & 0 \\ -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}. \tag{4.3}$$

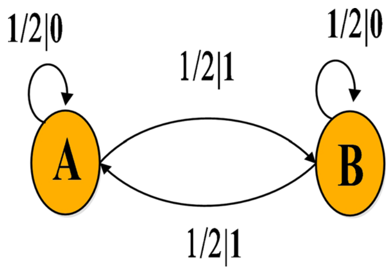

Fig. 3 Stochastic finite-state machine of GHZ state

Table 1 Probability of words for stochastic machine of GHZ state

Word w	Probability $Pr(w)$
0	1/2
1	1/2
00	1/4
01	1/4
10	1/4
11	1/4
000	1/8
001	1/8
010	1/8
011	1/8
100	1/8
101	1/8
110	1/8
111	1/8

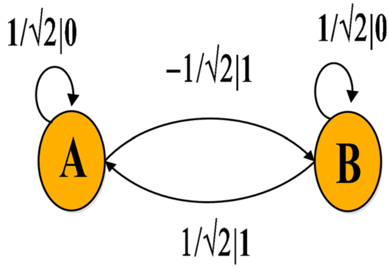

Fig. 4 Quantum finite-state machine of GHZ state

Table 2 Probability of words for quantum machine of GHZ state

Word w	Probability $Pr(w)$
0	1/2
1	1/2
00	1/4
01	1/4
10	1/4
11	1/4
000	1/8
001	1/8
010	1/8
011	1/8
100	1/8
101	1/8
110	1/8
111	1/8

Above, transition matrices are not stochastic. We normalized the matrices to form stochastic matrices such that $T^{(y)} = \langle T^{(y)}|T^{(y)}\rangle$. We encode the above three matrices of AKLT state into two matrices in order to reduce calculations. Correspondingly, the ratio of rank $\left(P^{(0)}\right)$: rank $\left(P^{(1)}\right)$ is $1:2$. Therefore, $|Q| \geq 3$ and it will have minimal three states. Firstly, we wish to design a SFSM for AKLT state that produces same probabilities of words as quantum version of machine and transition matrix is unistochastic. In Fig. 5, we have shown a four-state stochastic machine for AKLT state and have a unistochastic matrix.

$$T^{(0)} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, \quad T^{(1)} = \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

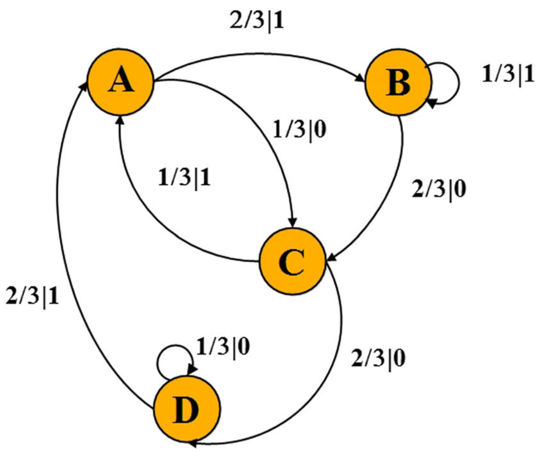

Fig. 5 Four-state stochastic finite-state machine of AKLT state

We have split the states $A,\ B$ to form four-state SFSM for AKLT state given as

$$S = \{A,\ B,\ C,\ D\},\quad Y = \{0,\ 1\},\quad T^{(1)} = \begin{pmatrix} 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \end{pmatrix},$$

$$T^{(0)} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{4.4}$$

It has a unistochastic matrix $T = T^{(0)} + T^{(1)}$.

$$T = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}. \tag{4.5}$$

Correspondingly, matrix $T(4.5)$ satisfies $T_{ij} = |U_{ij}|^2$, where U is unitary matrix (Fig. 6). Thus, the quantized version of AKLT state is given by

$$Q = \{A,\ B,\ C,\ D\},\quad Y = \{0,\ 1\},\quad U = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 & 0 & -\sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \tag{4.6}$$

$$P^{(0)} = |e_1\rangle\ \langle e_1| + |e_2\rangle\ \langle e_2|,\ P^{(1)} = |e_3\rangle\ \langle e_3| + |e_4\rangle\ \langle e_4|.$$

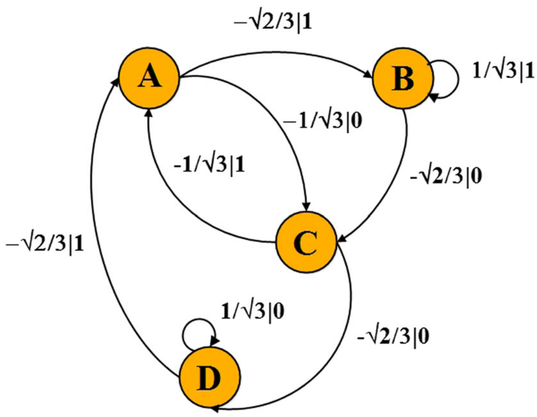

Fig. 6 Quantum finite-state machine of AKLT state

Table 3 Probability of words for QFSM of AKLT state

Word w	Probability $Pr(w)$
00	1/4
01	1/4
10	1/4
11	1/4
000	1/12
001	1/6
010	1/12
011	1/6
100	1/6
101	1/12
110	1/6
111	1/12

The probability of single letter $\Pr(0) = \Pr(1) = 1$, i.e., the ratio of rank $\left(P^{(0)}\right)$: rank $\left(P^{(1)}\right)$ is 1 : 1. The probability of words of QFSM for AKLT state with words of length $L = 2,\ 3$ is computed and is shown in Table 3.

4.3 Cluster state

Recently, cluster states have found widespread interest in quantum information theory. The reason behind this is measurement-based quantum computation, which is different from circuit model in quantum computation. Generally, cluster states are also graph states. We have designed a 2-state quantum finite-state machine of cluster state of 3-qubits:

$$|\phi\rangle_{c3} = |+\rangle_1|+\rangle_2|+\rangle_3 = \frac{1}{\sqrt{2}}\left((|+\rangle_1|0\rangle_2) + (|-\rangle_1|1\rangle_2)\right)(|0\rangle_3 + |1\rangle_3)$$
$$= \frac{1}{\sqrt{2}}(|+\rangle_1|0\rangle_2|+\rangle_3) + (|-\rangle_1|1\rangle_2|-\rangle_3) = |GHZ_3\rangle. \tag{4.7}$$

It shows that the cluster state $|\phi\rangle_{c3}$ of 3-qubits is equivalent to GHZ state. Therefore, its QFSM is similar to GHZ state of 3-qubits, which is shown in Fig. 4, and the probability of words is given in Table 2 under Sect. 4.1.

4.4 W state

It is easier to construct models with bistochastic matrices than to construct it with a unistochastic matrix. But, it is not sure that if we can find a stochastic model with a bistochastic matrix that will actually be unistochastic matrix. The four basis states named $(\varphi_A, \varphi_B, \varphi_C, \varphi_D)$ spanning the Hilbert space H for W state are

$$|\varphi_A\rangle = |1000\rangle, \quad |\varphi_B\rangle = |0100\rangle, \quad |\varphi_C\rangle = |0010\rangle, \quad |\varphi_D\rangle = |0001\rangle.$$

We constructed the 4-state quantum finite-state machine of 4-qubit W state by using the projection matrices 3.10 such that

$$Q = \{A, B, C, D\}, \quad Y = \{0, 1\}, \quad U = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \tag{4.8}$$

$$P(1)^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P(2)^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$P(3)^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P(4)^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P(1)^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P(2)^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P(3)^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P(4)^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

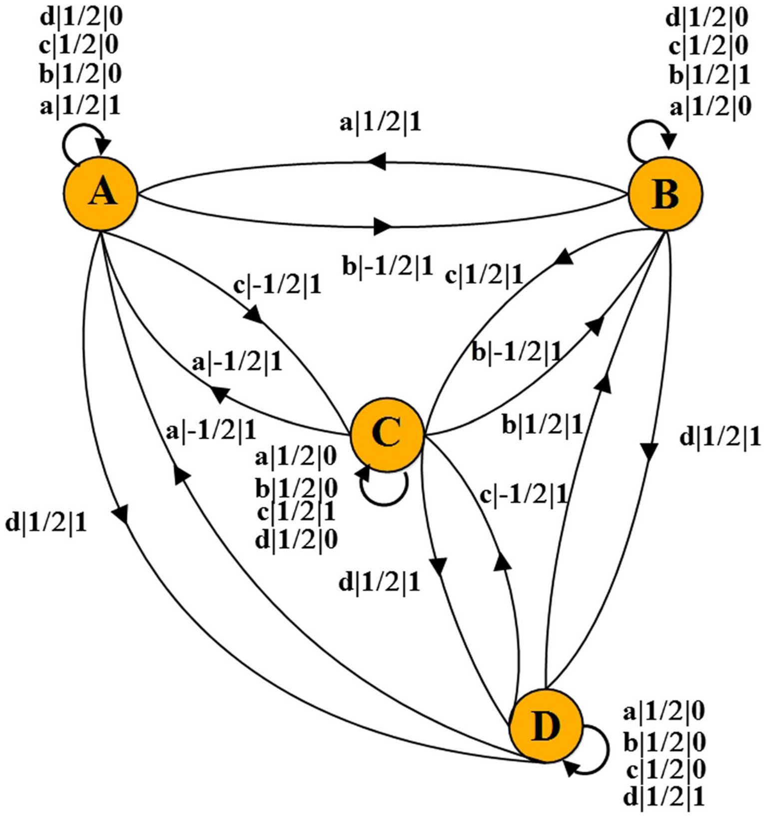

Fig. 7 Quantum finite-state machine of W state

The quantized version of W state shown in Fig. 7 represents unitary operations. Consider $X = \{a, b, c, d\}$, the input sequence $(abcd)^+$ and the probability of different word sequences of length $L = 4$, i.e., $w_1 w_2 w_3 w_4 \in Y^4$ is computed as $\Pr(w_L) = Tr\,(U^*(w_L)\rho U(w_L))$, where $\rho = \frac{1}{4} \cdot 1$.

$$UP(1)^0 = \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix}, \quad UP(1)^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$UP(2)^0 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 1 \end{pmatrix},$$

$$UP(2)^1 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad UP(3)^0 = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{pmatrix},$$

$$UP(3)^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$UP(4)^0 = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \quad UP(4)^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Quantum information has the ability to classify entanglement by means of some mathematical or physical uniformity. Therefore, it helps in increasing the practical abilities of quantum information protocols. In quantum information theory, many multiparticle entangled states (*GHZ* and *W* state) and metrology can be represented by MPS. Cluster states are featuring in the context of measurement-based quantum computing. MPS are so powerful and efficient such that they are optimally suited for quantum state tomography in condensed matter physics.

5 Conclusion

In this paper, we have efficiently simulated matrix product states with a broader quantum computational theory and investigated their relationship with quantum finite-state machine (QFSM) using unitary criteria. It has been proved that QFSM is equivalent to MPS representations of ground state of quantum spin systems. In future, we will design quantum version of projected entangled pair state (PEPS) representations of ground state of quantum spin systems.

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